Is Democracy Possible?

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March 30, 2012
What are we talking about?

A system of government by the whole population or all the eligible members of a state, typically through elected representatives.
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A system of government by the whole population or all the eligible members of a state, typically through elected representatives.

More generally, we’re talking about a specific form of group decision making —

- Deciding whether a building project should take place
- Deciding whether an amendment to a law should pass
- Choosing what/where to eat with a group of friends
What are we trying not to talk about?

- Why democracy is a good/bad idea
The process
The process
So what can go wrong?

- voting fraud - carousel voting, intimidation
  - statistical methods can sometimes be used to detect anomalies.
- counting fraud - particularly in automated voting machines
  - Verifying that the voting program works as desired; having source code is not enough.
  - Verifying the integrity of the data; encryption is not enough
  - If someone has physical access to the voting machine, it’s virtually impossible to secure.
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- But what about the voting system itself?
What is the point of democracy?

- Ensure “good” decisions are made
Ensure “good” decisions are made

Democracy is the recurrent suspicion that more than half of the people are right more than half the time.

– E.B. White
What is the point of democracy?

- Ensure “good” decisions are made
- Reflect the will of the people
What is the point of democracy?

- Ensure “good” decisions are made
- Reflect the will of the people
  - Which people? All of them?
  - What if 51% of people really don’t like the other 49%?
The purpose of voting is to obtain a collective preference (or social choice) from a set of individual preferences.

A preference is some sort of “goodness” ordering over outcomes

\[ \text{pizza} > _\text{nir} \text{ curry} > _\text{nir} \text{ stir fry} \]

\[ \text{pizza} > _\text{frank} \text{ stir fry} > _\text{frank} \text{ curry} \]
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\[ \text{pizza} >_{frank} \text{stir fry} >_{frank} \text{curry} \]

\[ \text{pizza} > \text{stir fry} = \text{curry} \]
7 people are trying to decide whether to eat Pizza or Chinese.

- 3 voters \( P > C > I \)
- 2 voters \( C > P > I \)
- 2 voters \( I > C > P \)

Chinese will win with 4 votes to 3.
7 people are trying to decide whether to eat Pizza or Chinese.

- 3 voters $P > C > I$
- 2 voters $C > P > I$
- 2 voters $I > C > P$

Chinese will win with 4 votes to 3.

If the choice of Indian is introduced, then pizza will win and Chinese will come second.

We’ve introduced an “irrelevant” alternative (as it still comes last) which has reversed the outcome.

This feels “unfair”
The following properties of voting systems are generally considered desirable:

**U**: Anyone can have any sort of consistent preference — anyone can vote for anything. This is known as the condition of *universal domain*.

**P**: If everyone voting prefers \( X \) to \( Y \), then in the result, \( X \) should be ranked more highly than \( Y \). This is the *weak Pareto principle*.

**D**: There is no individual such that no matter what anyone else prefers, they can decide on the outcome. This is the *non-dictatorship principle*. 
Properties of Voting Systems

The following properties of voting systems are generally considered desirable:

I: If a voting system combines two objects \( a, b \) so that \( a \geq b \) for a set of individuals who have different orderings (e.g. \( a \geq_1 b, b \geq_2 a, b \geq_3 a \)), then as long as these different orderings hold, the voting system will always result in \( a \geq b \).

In other words, \( a \)'s relation to \( c \) (and \( c \)'s to \( b \)) doesn't matter.

Example

\[
a \geq b \quad \text{if} \quad (acbd, dbac)
\]

Then

\[
(abcd, bdca)
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\[
(abcd, bacd)
\]
\[
(acdb, bcda)
\]
Properties of voting systems

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Can we find a voting system that satisfies all of these properties?
Properties of voting systems

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Can we find a voting system that satisfies all of these properties? NO!
So why do we care?

Given a finite number of individuals (even 2!), and at least three possibilities, there is no way to create a voting system for which conditions $U, P, D$ and $I$ hold.
Let’s assume we have $n$ people voting over possibilities $a, b, c, \ldots$.

- Let’s assume that for all individuals rank $a$ the highest, and $b$ the lowest.
- Since $a$ is preferred over every other outcome, by $P$ it must be ranked most highly.
- Similarly, $b$ is ranked as the least preferred outcome.
Now let's lift $b$ up for $R_1$ by 1 position.

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\begin{align*}
\begin{array}{cccccc}
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a & \ldots & a & a & a & \ldots & a & a \\
. & \ldots & . & . & . & \ldots & . & . \\
b & \ldots & . & . & . & \ldots & . & . \\
. & \ldots & b & b & b & \ldots & b & . \\
\end{array}
\end{align*}

Repeat until $b$ is $R_1$'s most preferred outcome.
\[
\begin{array}{cccccc}
R_1 & \ldots & R_{m-1} & R_m & R_{m+1} & \ldots & R_n & \text{outcome} \\
\hline
a & \ldots & a & a & a & \ldots & a & a \\
. & \ldots & . & . & . & \ldots & . & . \\
b & \ldots & . & . & . & \ldots & . & . \\
. & \ldots & b & b & b & \ldots & b & . \\
\end{array}
\]

Repeat until \( b \) is \( R_1 \)'s most preferred outcome.
Now since we've only actually reordered \( b \) and \( a \), by \( I \), \( a \) must be first or second in the outcomes. Let's assume it remains at the top. So we repeatedly raise \( b \) for the 2nd person, 3rd person etc, until \( b \) gets to the top. Let's say this happens for person \( m \). Note that if we end up doing this for all \( R \)'s, by \( P \) we're guaranteed to have \( b \) as the most preferred outcome, so this is always possible.
Now since we’ve only actually reordered $b$ and $a$, by $I$, $a$ must be first or second in the outcomes.

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So we repeatedly raise $b$ for the 2nd person, 3rd person etc, until $b$ gets to the top.

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Note that if we end up doing this for all $R$’s, by $P$ we’re guaranteed to have $b$ as the most preferred outcome, so this is always possible.
Again, since we’re only dealing with \( a \) and \( b \), by \( I \) this is the only outcome that should be affected.
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- Let’s move $a$ to the bottom for all $i < m$ and to the 2nd most preferred position for all $i > m$. 
Let’s move $a$ to the bottom for all $i < m$ and to the 2nd most preferred position for all $i > m$.

For the highlighted case, $b$ hasn’t moved with regards to anything else and must therefore be ranked most highly due to $l$.

Since $b$ was only exchanged with $a$ in the highlighted case, it cannot change ranking with anything other than $a$. So in the first situation, $b$ must rank highest apart from possibly $a$. 

$$\begin{array}{ccccccc|c}
R_1 & \ldots & R_{m-1} & R_m & R_{m+1} & \ldots & R_n & \text{outcome} \\
\hline
b & \ldots & b & a & \ldots & \ldots & \cdot & \\
. & \ldots & . & b & \ldots & \ldots & \cdot & \\
. & \ldots & . & . & a & \ldots & a & . \\
a & \ldots & a & . & b & \ldots & b & . \\
\end{array}$$

$$\begin{array}{ccccccc|c}
R_1 & \ldots & R_{m-1} & R_m & R_{m+1} & \ldots & R_n & \text{outcome} \\
\hline
b & \ldots & b & b & \ldots & \ldots & \cdot & b \\
. & \ldots & . & a & \ldots & \ldots & \cdot & . \\
. & \ldots & . & . & a & \ldots & a & . \\
a & \ldots & a & . & b & \ldots & b & . \\
\end{array}$$
So we know that in the case at the bottom, \( b \) must rank highest apart from possibly \( a \).

- Comparing, note that \( a \) and \( b \) haven’t moved w.r.t each other.
- So since \( b \) must rank highest in the bottom case apart form \( a \), \( a \) must rank highest in the bottom case.
So we know that in the case at the bottom, $b$ must rank highest apart from possibly $a$.

Comparing, note that $a$ and $b$ haven’t moved w.r.t each other.

So since $b$ must rank highest in the bottom case apart from form $a$, $a$ must rank highest in the bottom case.
We’ve shown that if \( a \) is ranked lowest for \( i < m \) and second lowest for \( m > i \) and highest for \( i = m \), \( a \) will be highest in the vote.
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- Let’s switch the rankings of $a$ and $b$ for $i > m$.
- Can $b$ move above $a$ in the outcomes?
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- Let's switch the rankings of $a$ and $b$ for $i > m$.
- Can $b$ move above $a$ in the outcomes?
- No as $c > b$ so by $P$ $c$ has to rank above $b$.
- Therefore $a$ remains at the top, and $c$ ranks above $b$. 
Create an arbitrary set of profiles, except for $R_m$ for who $a > b$.

$I$ means that $c$ can’t have an effect on the rankings of $a$ and $b$.

The rankings between $a$ and $c$ are as in the previous step (i.e. $c > a$ except for $R_m$) — by $I$ $a$ must remain preferred over $c$.

$c$ is above $b$ so by $P$ it is preferred.

So $a > c$ and $c > b$ so $a > b$ whenever $a >_{R_m} b$

In other words, $R_m$ is a dictator for choice $a$. 

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Could we have different dictators for different choices (e.g. one for $a$, a different one for $b$ etc)?
Could we have different dictators for different choices (e.g. one for $a$, a different one for $b$ etc)?

No; as what would happen when both dictators try exert their power?

We have used $I$, $P$ and $U$ to show that $D$ cannot hold.

No voting system can satisfy all of the desired conditions simultaneously!
Is it all bad news?

- So no voting system is perfect.
- But we could lift one of the requirements.
In some situations, it is possible to constrain the types of preferences individuals can have.

For example, selecting the volume of music for a party.

It’s been shown that in such situations, majority rule voting works.
Not requiring $P$ is not as useful; it has been shown that either a dictator still exists, or an *inverse dictator*.

For an inverse dictator, if $a >_i b$ then $b > a$. 
If we lift $I$, then as seen in FPTP, voting for an “irrelevant” alternative can affect the outcome.

This means that a voter could change the winner by voting for someone that they do not really want to vote for.

In other words, strategic voting is a necessary feature of any voting system which ignores $I$. This include FPTP, AV, Borda and most other “widely used” voting systems.
Strategic voting means a voter must consider all the other voter’s choices when making their choice.

“If a votes x then I should vote y. But if a thinks I’ll vote y, they’ll vote z, in which case I should vote x, …”

Voting becomes a *game theoretic* problem.

Solving game theoretic problems can be hard:

- Strategic voting could mean an unexpected (and unwanted) outcome.
- But computing an optimal voting strategy could be very difficult, disincentivising such behaviour.
And another thing...

- Note that we only spoke about 3 or more alternatives.
- What if we’ve only got 2? Then Arrow’s theorem doesn’t hold.
- So we could vote on 2 issues.
- Why not always limit to 2 alternatives (e.g. if there are 4 alternatives, pit 2 of them against each other in two “preliminary rounds”) and then have the winners fight it out?
- The order in which the alternatives are given alters the final outcome.
So where are we?

- The voting process is vulnerable at various points
  - Social, political and technical vulnerabilities occur when running elections.
  - Mathematical vulnerabilities appear when trying to create a fair voting mechanism.
- The latter result indicates that strategic voting is always possible.
- But what if, instead of trying to find a perfect voting mechanism, voters could change their preferences?
  - Perhaps access to better explanations about outcomes of decisions could align people’s preferences?
  - If so, increasing debate, participative democracy etc, might be the best way to make democracy work.